Quantum Nonlocality

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first game

THE MAGIC SQUARE GAME*

* P.K. Aravind (2004)

Magic Square

A magic square is a 3×3 matrix with elements in {0, 1} that satisfies:

Row Rule :

0	0	0	even
0	1	1	even
1	1	0	even

Column Rule :

0	0	1
0	1	1
1	0	1
odd	odd	odd

Can a Magic Square exist?

0	0	0
0	1	1
1	0	1 ?

0	0	0
0	0	0
1	1	?

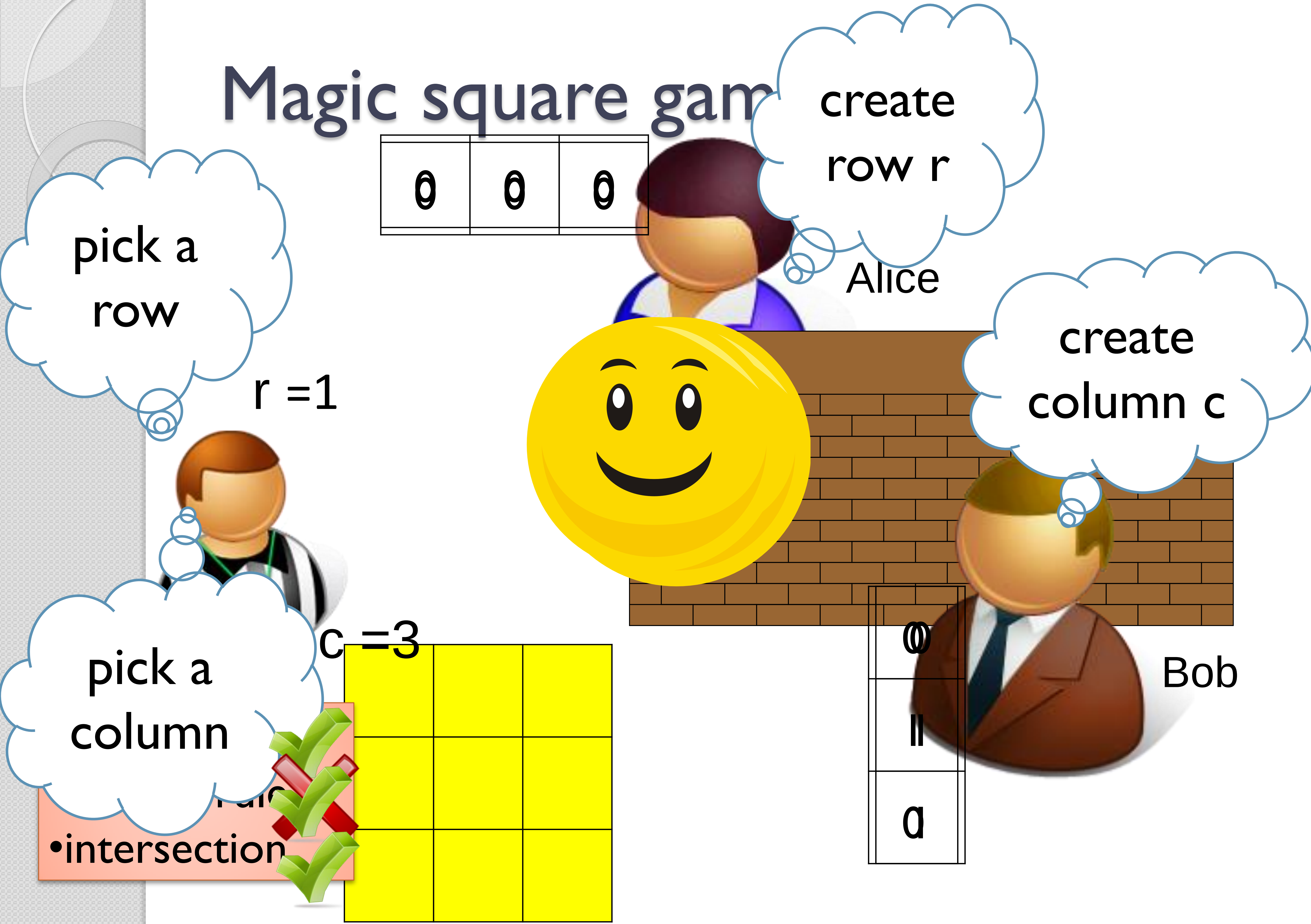
a	b	c
d	e	f
g	h	i

$a+b+c+d+e+f+g+h+i$ is even (row rule)

$a+d+g+b+e+h+c+f+i$ is odd (column rule)

CONTRADICTION!

Magic square gam

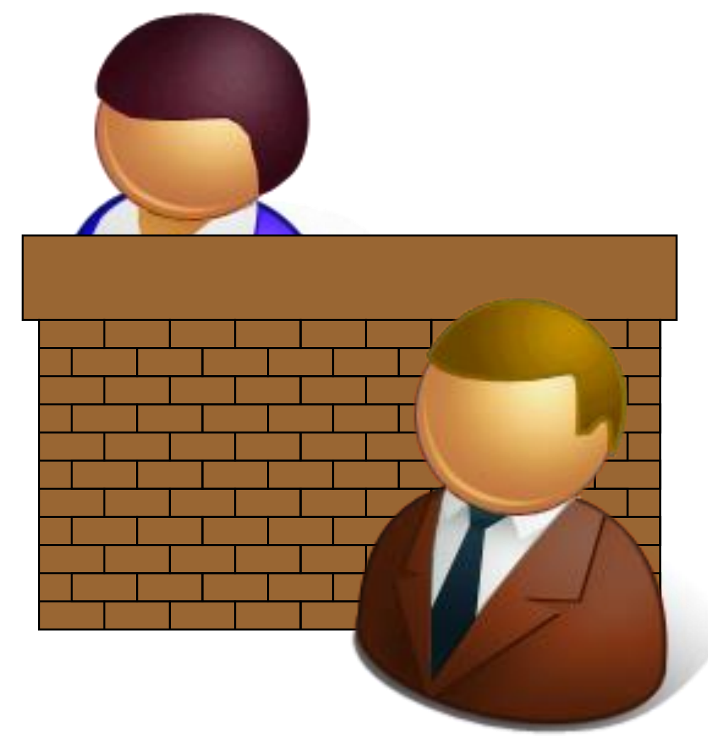


Success probability

- Let w(G) be the maximum success probability for Alice and Bob who play a game G, and where the questions are asked uniformly at random.
- Do Alice et Bob have a winning strategy for the Magic Square game? (w(G) = 1?)

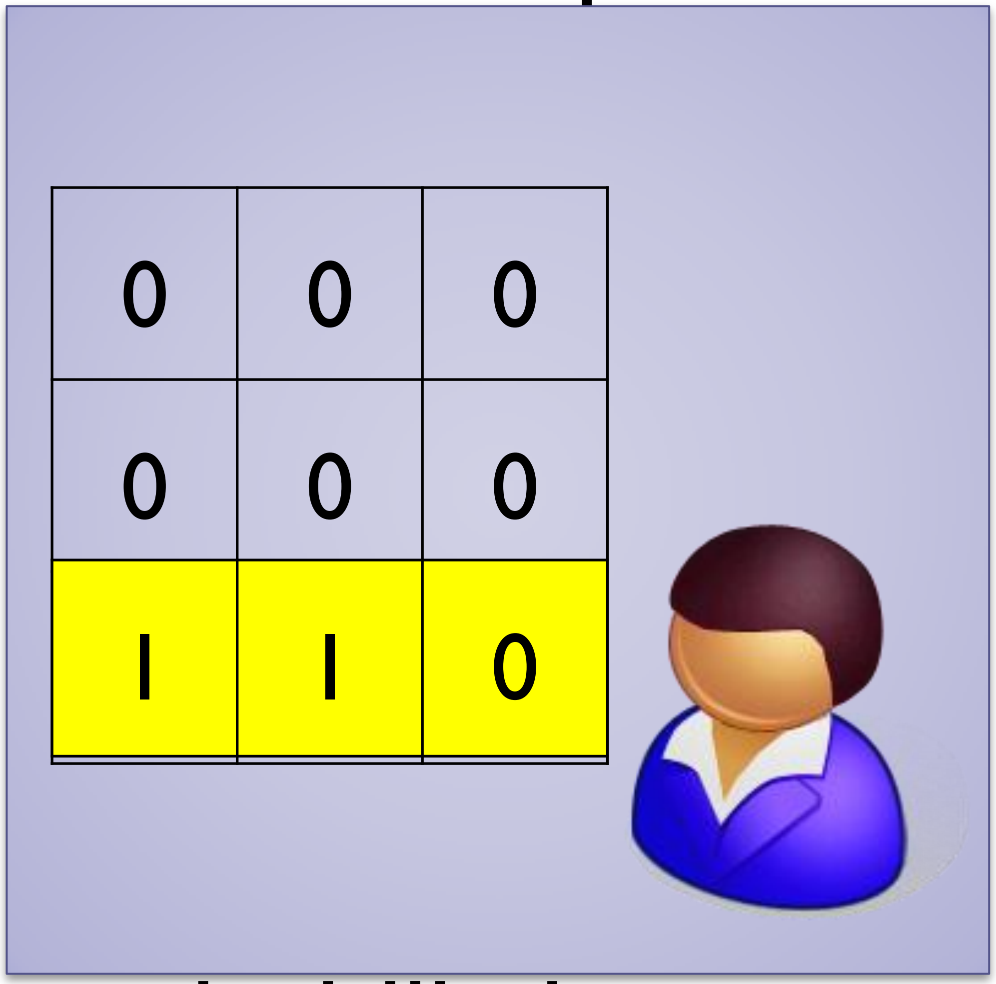
- A classical winning strategy corresponds to a magic square, hence $w(G) \neq 1$.

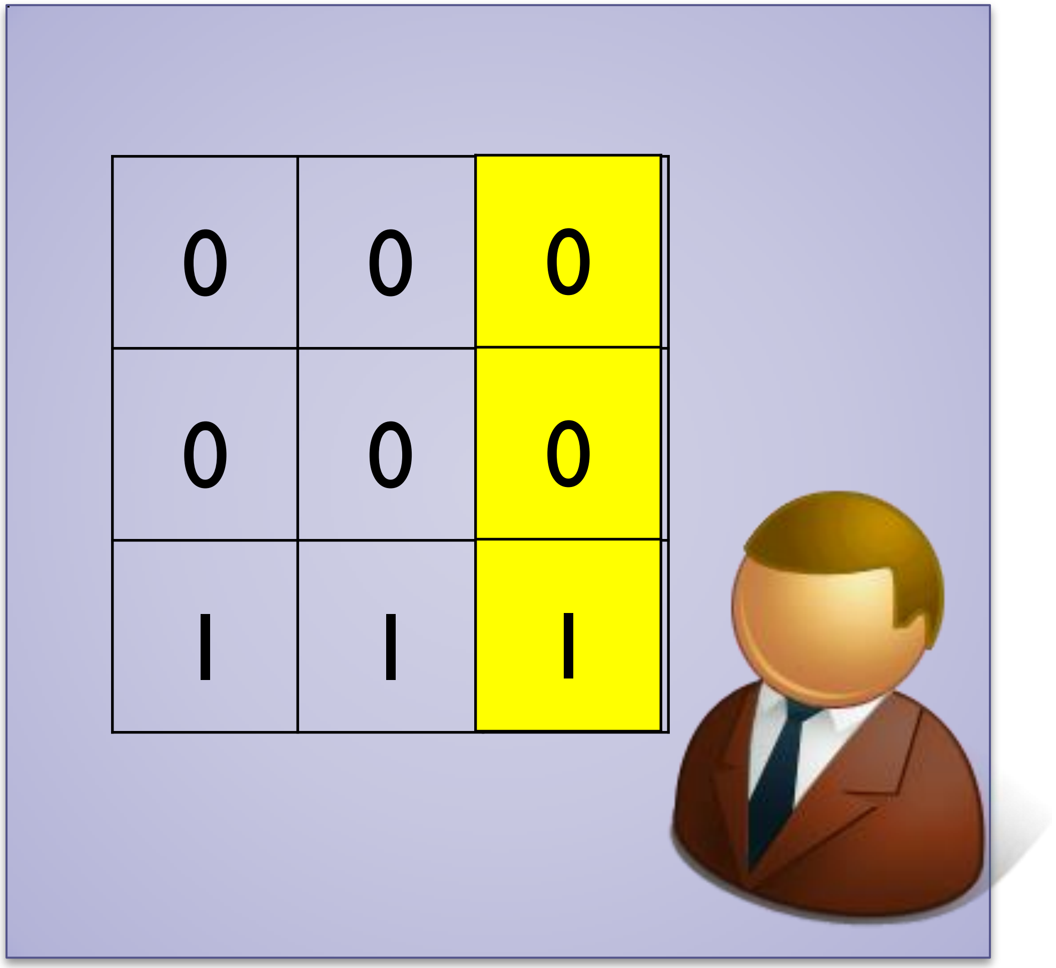
- row rule
- column rule
- intersection



Maximum success probability

- The best deterministic strategy wins on 8 out of the 9 possible questions:





- A probabilistic strategy is a probability distribution on deterministic strategies:
  - $w(G) = 8/9$

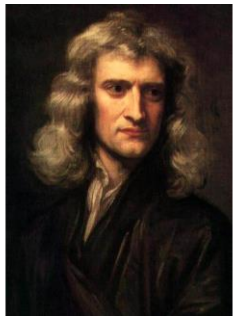
What if?

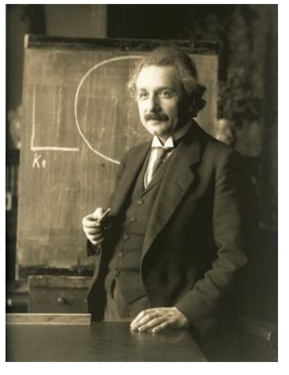
What would you think if you were the referee and you saw that Alice and Bob were always winning?

- Are they cheating?
- Are they being really lucky?

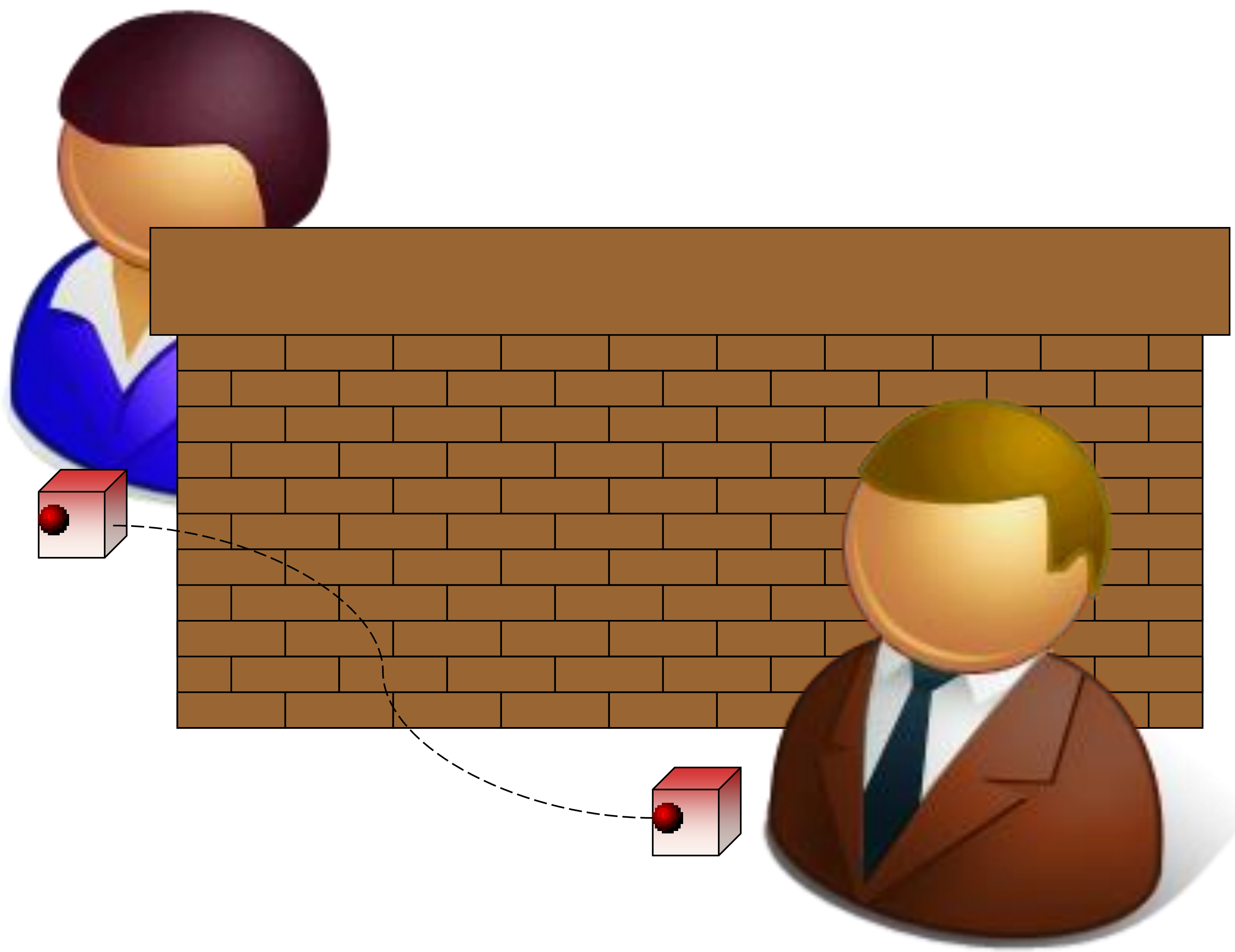
$$P(\text{win all } k \text{ repetition}) \leq \frac{8}{9}^{k}$$

- Do the laws of physics forbid this?





An explanation



Quantum Winning Strategy

Starting with

$$|\psi\rangle = \frac{1}{2}|0011\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle + \frac{1}{2}|1100\rangle$$

Alice applies transformation A_r *and Bob applies transformation* B_c *:*

$$A_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix},\ A_2 = \frac{1}{2}\begin{bmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{bmatrix},\ A_3 = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$B_1 = \frac{1}{2}\begin{bmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{bmatrix},\ B_2 = \frac{1}{2}\begin{bmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{bmatrix},\ B_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

For instance, if *r=1* and *c=2,* the result will be:

$$|\psi'\rangle = \frac{1}{\sqrt{8}}(|0000\rangle + |0001\rangle + i|0110\rangle + i|0111\rangle - i|1000\rangle - i|1001\rangle + i|1110\rangle - i|1111\rangle)$$

Magic Square game

- if *r=1* and *c=2*,

$$|\psi'\rangle = \frac{1}{\sqrt{8}}(|0000\rangle + |0001\rangle + i\,|0110\rangle + i\,|0111\rangle - i\,|1000\rangle - i\,|1001\rangle + i\,|1110\rangle - i\,|1111\rangle)$$

- Alice and Bob measure to get:

Alice	Bob	Probability
00	00	1/8
00	01	1/8
01	10	1/8
01	11	1/8
10	00	1/8
10	01	1/8
11	10	1/8
11	11	1/8

Alice	Bob	Probability
000	001	1/8
000	010	1/8
011	100	1/8
011	111	1/8
101	001	1/8
101	010	1/8
110	100	1/8
110	111	1/8

They compute their third bit in order to satisfy the row or column rule, respectively.

Magic Square game

So for *r=1* and *c=2,* Alice and Bob appear to be sharing one of the following partial magic squares, with equal probability:

0	0	0
	0	
	1	

0	0	0
	1	
	0	

0	1	1
	0	
	0	

0	1	1
	1	
	1	

1	0	1
	0	
	1	

1	0	1
	1	
	0	

1	1	0
	0	
	0	

1	1	0
	1	
	1	

Review of Quantum Winning Strategy

- Alice and Bob share the entangled state

$$|\psi\rangle = \frac{1}{2}|0011\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle + \frac{1}{2}|1100\rangle$$

- Alice applies transformation A_r *and Bob applies transformation* B_c *(*depending on their input).

$$A_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix},\ A_2 = \frac{1}{2}\begin{bmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{bmatrix},\ A_3 = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$B_1 = \frac{1}{2}\begin{bmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{bmatrix},\ B_2 = \frac{1}{2}\begin{bmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{bmatrix},\ B_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

- Alice et Bob mesure their system and obtain 2 bits each.
- The third bit is computed so that the row rule, or column rule (whichever is the case) is satisfied.

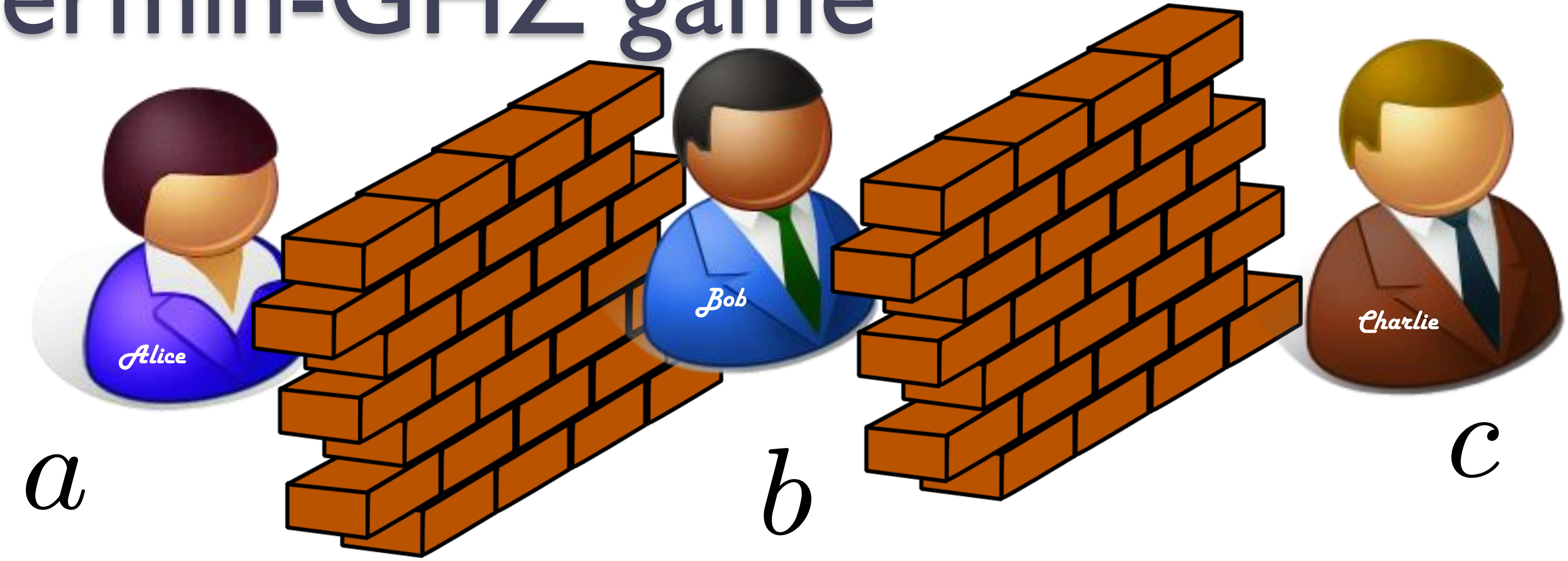
This is a winning strategy

second game

THE GHZ* GAME

*Greenberger-Horne-Zeilinger (1989), Mermin (1990)

Mermin-GHZ game



$$a, b, c \in \{0, 1\}$$

$$x, y, z \in \{0, 1\}$$

promise:
$x + y + z$
is even

winning condition

$\frac{x+y+z}{2}$ is even $\Rightarrow a + b + c$ is even

$\frac{x+y+z}{2}$ is odd $\Rightarrow a + b + c$ is odd

A winning strategy?

winning condition

$\frac{x+y+z}{2}$ is even $\Rightarrow a+b+c$ is even

$\frac{x+y+z}{2}$ is odd $\Rightarrow a+b+c$ is odd

best success probability: ¾

Input (x,y,z)	Winning outputs (a,b,c)
(0,0,0) : even	(0,0,0),(0,1,1), (1,0,1), (1,1,0) : even
(0,1,1) : odd	(0,0,1),(0,1,0), (1,0,0), (1,1,1) : odd
(1,0,1) : odd	(0,0,1),(0,1,0), (1,0,0), (1,1,1) : odd
(1,1,0) : odd	(0,0,1),(0,1,0), (1,0,0), (1,1,1) : odd

general deterministic strategy:

A: $0 \rightarrow a_0, 1 \rightarrow a_1$

B: $0 \rightarrow b_0, 1 \rightarrow b_1$

C: $0 \rightarrow c_0, 1 \rightarrow c_1$

a winning strategy must satisfy:

$a_0 + b_0 + c_0 =$even

$a_0 + b_1 + c_1 =$odd

$a_1 + b_0 + c_1 =$odd

$a_1 + b_1 + c_0 =$odd

even odd

CONTRADICTION!

Quantum Winning Strategy

The player's strategy is to share the entangled state: $\frac{1}{\sqrt{2}}|0_A 0_B 0_C\rangle + \frac{1}{\sqrt{2}}|1_A 1_B 1_C\rangle$

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle \neq (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha'|0\rangle + \beta'|1\rangle) \otimes (\alpha''|0\rangle + \beta''|1\rangle)$$

Quantum winning strategy

Starting with $|\psi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$, each player does the following:

1. If the player's input is 1, apply the unitary: transformation:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

result: $\frac{1}{\sqrt{2}}|0_A 0_B 0_C\rangle + \frac{1}{\sqrt{2}} i^m |1_A 1_B 1_C\rangle$

2. Apply the Hadamard transform.

result: $\begin{cases} \frac{1}{2}|0_A 0_B 0_C\rangle + \frac{1}{2}|0_A 1_B 1_C\rangle + \frac{1}{2}|1_A 0_B 1_C\rangle + \frac{1}{2}|1_A 1_B 0_C\rangle, m/2 \text{ is even} \\ \frac{1}{2}|0_A 0_B 1_C\rangle + \frac{1}{2}|0_A 1_B 0_C\rangle + \frac{1}{2}|1_A 0_B 0_C\rangle + \frac{1}{2}|1_A 1_B 1_C\rangle, m/2 \text{ is odd} \end{cases}$

3. Measure, and output the result to the referee.

winning condition

$\frac{x+y+z}{2}$ is even $\Rightarrow a+b+c$ is even

$\frac{x+y+z}{2}$ is odd $\Rightarrow a+b+c$ is odd

Pseudotelepathy

- Any classical winning strategy would require communication
- Quantum winning strategy achieved *without communication*
- Called *pseudotelepathy* or *nonlocal games*

Bell's Theorem (1964): "The predictions of quantum mechanics are incompatible with any local physical theory"

Local theory: $p(ab \mid xy, \lambda) = p(a \mid x, \lambda) p(b \mid y, \lambda)$